Lecture 2
The Wess-Zumino Model

Outline

- Review: two component spinors.
- The simplest SUSY Lagrangian: the free Wess-Zumino model.
- The SUSY algebra and the off-shell formalism.
- Noether theorem for SUSY: the supercurrent.
- Interactions in the WZ-model: the superpotential.

Reading: Terning 2.1-2.4, A.1-2

Two component spinors

The massless Dirac Lagrangian, in four-component and two-component forms:

$$\mathcal{L} = i\overline{\psi}\gamma^\mu \partial_\mu \psi = i\psi_L^\dagger \overline{\sigma}^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R$$

with

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} , \quad \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} , \quad \overline{\psi} = \psi^\dagger \gamma^0 = \begin{pmatrix} \psi_R^\dagger & \psi_L^\dagger \end{pmatrix}$$

Convention: focus on the L-component. In other words: the index “L” is always implied.

Aside: Lorentz-transformations

Under Lorentz transformation (with rotation angles $\vec{\theta}$ and boost parameters $\vec{\beta}$), the "L" and "R" helicities do not mix:

$$\begin{aligned}
\psi_L &\rightarrow (1 - i\vec{\theta}\cdot\tfrac{\vec{\sigma}}{2} - \vec{\beta}\cdot\tfrac{\vec{\sigma}}{2})\psi_L \\
\psi_R &\rightarrow (1 - i\vec{\theta}\cdot\tfrac{\vec{\sigma}}{2} + \vec{\beta}\cdot\tfrac{\vec{\sigma}}{2})\psi_R
\end{aligned}$$

A simple Lorentz invariant (the Majorana mass term):

$$\chi_L^T(-i\sigma_2)\psi_L = -\chi_\alpha\epsilon^{\alpha\beta}\psi_\beta = \chi^\beta\psi_\beta = \chi\psi$$

Check: the transposed spinor transforms as

$$\psi_L^T \rightarrow \psi_L^T(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}^T}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}^T}{2}) = \psi_L^T\sigma_2(1 + i\vec{\theta}\cdot\tfrac{\vec{\sigma}}{2} + \vec{\beta}\cdot\tfrac{\vec{\sigma}}{2})\sigma_2$$

Notation:

$$\begin{aligned}
\psi_\alpha &= \epsilon_{\alpha\beta}\psi^\beta \;, \quad \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \\
\boldsymbol{\epsilon}_{\alpha\beta} &= -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \;, \quad \boldsymbol{\epsilon}^{\alpha\beta} = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\end{aligned}$$

Signs: careful with ordering of indices, ordering of fermions, upper/lower indices...

Sample manipulation:

$$\chi\psi = \chi^\alpha \psi_\alpha = \chi^\alpha \epsilon_{\alpha\beta} \psi^\beta = -\epsilon_{\alpha\beta} \psi^\beta \chi^\alpha = \psi^\beta \epsilon_{\beta\alpha} \chi^\alpha = \psi\chi$$

The complex conjugate of the "L"-spinor transforms like ψ_R:

$$\sigma_2 \psi_L^* \quad \rightarrow \quad \sigma_2 (1 + i\vec{\theta} \cdot \tfrac{\vec{\sigma}^*}{2} - \vec{\beta} \cdot \tfrac{\vec{\sigma}^*}{2}) \psi_L^* = (1 - i\vec{\theta} \cdot \tfrac{\vec{\sigma}}{2} + \vec{\beta} \cdot \tfrac{\vec{\sigma}}{2}) \sigma_2 \psi_L^*$$

Example: since $\chi_R^\dagger \sim \psi_L^T \sigma_2$ (as far as Lorentz is concerned), the Majorana mass term (above) is equivalent to

$$\chi_R^\dagger \psi_L = \chi^\alpha \psi_\alpha$$

Conjugate spinors have dotted indices with "SW to NE" contraction

$$\psi_L^\dagger \chi_R = \psi_{\dot{\alpha}} \chi^{\dot{\alpha}}$$

The Pauli matrices have transformation properties according to the index structure

$$\sigma^{\mu}_{\alpha\dot{\alpha}} \ , \quad \overline{\sigma}^{\mu\dot{\alpha}\alpha}$$

Example: a Lorentz-vector is formed from two-component spinors by the combination:

$$\chi^{\dagger}\overline{\sigma}^{\mu}\psi = \chi^{\dagger}_{\dot{\alpha}}\overline{\sigma}^{\mu\dot{\alpha}\alpha}\psi_{\alpha}$$

The free Wess–Zumino model

The simplest representation of the superalgebra: the massless chiral supermultiplet.

Particle content: a massless complex scalar field and a massless two component fermion (a Weyl fermion).

Goal: present a field theory with this particle content, and show that it realizes the superalgebra.

Proposal:

$$S = \int d^4x \ (\mathcal{L}_{\rm s} + \mathcal{L}_{\rm f})$$

where

$$\mathcal{L}_{\rm s} = \partial^\mu \phi^* \partial_\mu \phi \ , \qquad \mathcal{L}_{\rm f} = i\psi^\dagger \overline{\sigma}^\mu \partial_\mu \psi.$$

Convention for spacetime metric:

$$g^{\mu\nu} = \eta^{\mu\nu} = {\rm diag}(1, -1, -1, -1)$$

The SUSY transformation

Generic infinitesimal symmetry transformation

$$\phi \to \phi + \delta\phi$$
$$\psi \to \psi + \delta\psi$$

SUSY changes bosons into fermions so we consider the transformation

$$\delta\phi \quad = \quad \epsilon^\alpha \psi_\alpha = \epsilon\psi$$

Remarks:

- ϵ^α is an infinitesimal parameter with spinorial indices.
- Mass dimensions: $\dim(\phi) = 1$, $\dim(\psi) = \frac{3}{2} \Rightarrow \dim(\epsilon) = -\frac{1}{2}$.

SUSY changes fermions into bosons. Since the SUSY transformation parameter ϵ^α has dimension $(-\frac{1}{2})$, the transformation must include a derivative (recall $\dim(\partial) = 1$). Lorentz invariance determines the form as

$$\delta\psi_\alpha = -i(\sigma^\nu \epsilon^\dagger)_\alpha \, \partial_\nu \phi$$

SUSY of free WZ-model

The variation of the scalar Lagrangian under $\delta\phi = \epsilon\psi$:

$$\delta\mathcal{L}_{\rm s} = \partial^\mu \delta\phi \partial_\mu \phi^* + \partial^\mu \phi \partial_\mu \delta\phi^* = \epsilon \partial^\mu \psi \, \partial_\mu \phi^* + \epsilon^\dagger \partial^\mu \psi^\dagger \, \partial_\mu \phi$$

The variation of the fermion Lagrangian under

$$\delta\psi_\alpha = -i(\sigma^\nu \epsilon^\dagger)_\alpha \, \partial_\nu \phi \; , \quad \delta\psi^\dagger_{\dot\alpha} = i(\epsilon\sigma^\nu)_{\dot\alpha} \, \partial_\nu \phi^*$$

gives

$$\begin{aligned} \delta\mathcal{L}_{\rm f} &= i\delta\psi^\dagger \overline{\sigma}^\mu \partial_\mu \psi + i\psi^\dagger \overline{\sigma}^\mu \partial_\mu \delta\psi \\ &= -\epsilon\sigma^\nu \partial_\nu \phi^* \overline{\sigma}^\mu \partial_\mu \psi \; + \psi^\dagger \overline{\sigma}^\mu \sigma^\nu \epsilon^\dagger \, \partial_\mu \partial_\nu \phi \\ &= -\epsilon \partial^\mu \psi \, \partial_\mu \phi^* - \epsilon^\dagger \partial^\mu \psi^\dagger \, \partial_\mu \phi \\ &\quad + \partial_\mu \left(\epsilon \sigma^\mu \overline{\sigma}^\nu \psi \, \partial_\nu \phi^* - \epsilon\psi \, \partial^\mu \phi^* + \epsilon^\dagger \psi^\dagger \, \partial^\mu \phi \right) . \end{aligned}$$

The sum of the two variations is a total derivative so the action is invariant:

$$\delta S = 0$$

Remark: in the manipulations we needed the Pauli identities:

$$\left[\sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu\right]_\alpha^{\ \beta} = 2\eta^{\mu\nu}\delta_\alpha^\beta$$
$$\left[\overline{\sigma}^\mu\sigma^\nu + \overline{\sigma}^\nu\sigma^\mu\right]_{\dot\alpha}^{\ \dot\beta} = 2\eta^{\mu\nu}\delta_{\dot\alpha}^{\dot\beta}$$

These are the two component versions of the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

SUSY Commutators

Consistency: the commutator of two SUSY transformations must itself be a symmetry transformation. We need to show that it is in fact a translation, as the SUSY algebra indicates.

The commutator, acting on the complex scalar:

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\phi = -i(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger)\,\partial_\mu\phi$$

The commutator, acting on the fermion:

$$\begin{aligned}(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\psi_\alpha &= -i(\sigma^\nu\epsilon_1^\dagger)_\alpha\,\epsilon_2\partial_\nu\psi + i(\sigma^\nu\epsilon_2^\dagger)_\alpha\,\epsilon_1\partial_\nu\psi \\ &= -i(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger)\,\partial_\mu\psi_\alpha \\ &\quad +i(\epsilon_{1\alpha}\,\epsilon_2^\dagger\overline{\sigma}^\mu\partial_\mu\psi - \epsilon_{2\alpha}\,\epsilon_1^\dagger\overline{\sigma}^\mu\partial_\mu\psi).\end{aligned}$$

(Reorganized using the Fierz identity: $\chi_\alpha\,(\xi\eta) = -\xi_\alpha\,(\chi\eta) - (\xi\chi)\eta_\alpha$)

The last term vanishes upon imposing the fermion equation of motion. With this caveat, the commutator is a translation, with details the same for the two fields. Thus the SUSY algebra closes on-shell.

Counting Degrees of Freedom

The fermion e.o.m. projects out half of the degrees of freedom:

$$\overline{\sigma}^{\mu} p_{\mu} \psi = \begin{pmatrix} 0 & 0 \\ 0 & 2p \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} , \qquad p_{\mu} = (p, 0, 0, p)$$

Counting degrees of freedom shows that SUSY is not manifest off-shell:

	off-shell	on-shell
ϕ, ϕ^*	2 d.o.f.	2 d.o.f.
$\psi_{\alpha}, \psi^{\dagger}_{\dot{\alpha}}$	4 d.o.f.	2 d.o.f.

Restore SUSY off-shell: add an auxiliary boson field $\mathcal{F}$ with Lagrangian

$$\mathcal{L}_{\text{aux}} = \mathcal{F}^* \mathcal{F}$$

Recount degrees of freedom

	off-shell	on-shell
$\mathcal{F}, \mathcal{F}^*$	2 d.o.f.	0 d.o.f.

Maintain SUSY off-shell: transform the auxiliary field, and modify the transformation of the fermion

$$\begin{aligned}
\delta\mathcal{F} &= -i\epsilon^\dagger\overline{\sigma}^\mu\partial_\mu\psi \ , \quad \delta\mathcal{F}^* = i\partial_\mu\psi^\dagger\overline{\sigma}^\mu\epsilon \\
\delta\psi_\alpha &= -i(\sigma^\nu\epsilon^\dagger)_\alpha\,\partial_\nu\phi + \epsilon_\alpha\mathcal{F} \ , \quad \delta\psi^\dagger_{\dot\alpha} = +i(\epsilon\sigma^\nu)_{\dot\alpha}\,\partial_\nu\phi^* + \epsilon^\dagger_{\dot\alpha}\mathcal{F}^*
\end{aligned}$$

Transformation of the Lagrangian:

$$\begin{aligned}
\delta\mathcal{L}_{\rm aux} &= i\partial_\mu\psi^\dagger\overline{\sigma}^\mu\epsilon\,\mathcal{F} - i\epsilon^\dagger\overline{\sigma}^\mu\partial_\mu\psi\,\mathcal{F}^* \\
\delta^{\rm new}\mathcal{L}_f &= \delta^{\rm old}\mathcal{L}_f + i\epsilon^\dagger\overline{\sigma}^\mu\partial_\mu\psi\mathcal{F}^* + i\psi^\dagger\overline{\sigma}^\mu\partial_\mu(\epsilon\mathcal{F}) \\
&= \delta^{\rm old}\mathcal{L}_f + i\epsilon^\dagger\overline{\sigma}^\mu\partial_\mu\psi\mathcal{F}^* - i(\partial_\mu\psi^\dagger)\overline{\sigma}^\mu\epsilon\mathcal{F} + \partial_\mu(i\psi^\dagger\overline{\sigma}^\mu\epsilon\mathcal{F})
\end{aligned}$$

The last term is a total derivative so the action

$$S^{\rm new} = \int d^4x\,\mathcal{L}_{\rm free} = \int d^4x\,(\mathcal{L}_{\rm s} + \mathcal{L}_{\rm f} + \mathcal{L}_{\rm aux})$$

is invariant under SUSY transformations:

$$\delta S^{\rm new} = 0$$

Off-shell SUSY commutator

The previous computation, without using e.o.m.

$$
\begin{aligned}
(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\psi_\alpha \quad &= \quad -i(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger)\,\partial_\mu\psi_\alpha \\
&\quad\quad +i(\epsilon_{1\alpha}\,\epsilon_2^\dagger\overline{\sigma}^\mu\partial_\mu\psi - \epsilon_{2\alpha}\,\epsilon_1^\dagger\overline{\sigma}^\mu\partial_\mu\psi) \\
&\quad\quad +\delta_{\epsilon_2}\epsilon_{1\alpha}\mathcal{F} - \delta_{\epsilon_1}\epsilon_{2\alpha}\mathcal{F} \\
&= \quad -i(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger)\,\partial_\mu\psi_\alpha
\end{aligned}
$$

The point: the additional term in the SUSY transformation of the fermion, the depending on the auxiliary field, is precisely such that the last two lines cancel.

Conclusion: the SUSY algebra closes for off-shell fermions.

Off-shell SUSY commutator II

Issue: the commutator of SUSY transformations must also close when acting on the auxiliary field.

Computation:

$$\begin{aligned}
(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\mathcal{F} &= \delta_{\epsilon_2}(-i\epsilon_1^\dagger\overline{\sigma}^\mu\partial_\mu\psi) - \delta_{\epsilon_1}(-i\epsilon_2^\dagger\overline{\sigma}^\mu\partial_\mu\psi) \\
&= -i\epsilon_1^\dagger\overline{\sigma}^\mu\partial_\mu(-i\sigma^\nu\epsilon_2^\dagger\,\partial_\nu\phi + \epsilon_2\mathcal{F}) \\
&\quad +i\epsilon_2^\dagger\overline{\sigma}^\mu\partial_\mu(-i\sigma^\nu\epsilon_1^\dagger\,\partial_\nu\phi + \epsilon_1\mathcal{F}) \\
&= -i(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger)\,\partial_\mu\mathcal{F} \\
&\quad -\epsilon_1^\dagger\overline{\sigma}^\mu\sigma^\nu\epsilon_2^\dagger\,\partial_\mu\partial_\nu\phi + \epsilon_2^\dagger\overline{\sigma}^\mu\sigma^\nu\epsilon_1^\dagger\,\partial_\mu\partial_\nu\phi \\
&= -i(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger)\,\partial_\mu\mathcal{F}
\end{aligned}$$

Conclusion: the SUSY algebra closes

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})X = -i(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger)\,\partial_\mu X$$

for all the fields in the off-shell supermultiplet

$$X = \phi, \phi^*, \psi, \psi^\dagger, \mathcal{F}, \mathcal{F}^*$$

Noether's Theorem

Noether's theorem: corresponding to every continuous symmetry, there is a conserved current.

An infinitesimal transformation $X \to X + \delta X$ of the field X that leaves the action invariant, transforms the Lagrangian to a total derivative:

$$\delta\mathcal{L} = \mathcal{L}(X + \delta X) - \mathcal{L}(X) = \partial_\mu V^\mu$$

Identification of conserved current:

$$\begin{aligned} \partial_\mu V^\mu = \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial X}\delta X + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu X)}\right)\delta(\partial_\mu X) \\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu X)}\delta X\right) \end{aligned}$$

$$\Rightarrow \quad \epsilon\partial_\mu J^\mu = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu X)}\delta X - V^\mu\right) = 0$$

Ingredient: the equation of motion

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu X)}\right) = \frac{\partial\mathcal{L}}{\partial X}$$

The Supercurrent

The conserved supercurrent, J^μ_α:

$$
\begin{aligned}
\epsilon J^\mu + \epsilon^\dagger J^{\dagger\mu} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu X)}\,\delta X - V^\mu \\
&= \delta\phi\partial^\mu\phi^* + \delta\phi^*\partial^\mu\phi + i\psi^\dagger\overline{\sigma}^\mu\delta\psi - V^\mu \\
&= \epsilon\psi\partial^\mu\phi^* + \epsilon^\dagger\psi^\dagger\partial^\mu\phi + i\psi^\dagger\overline{\sigma}^\mu(-i\sigma^\nu\epsilon^\dagger\,\partial_\nu\phi + \epsilon\mathcal{F}) \\
&\quad -\epsilon\sigma^\mu\overline{\sigma}^\nu\psi\,\partial_\nu\phi^* + \epsilon\psi\,\partial^\mu\phi^* - \epsilon^\dagger\psi^\dagger\,\partial^\mu\phi - i\psi^\dagger\overline{\sigma}^\mu\epsilon\mathcal{F} \\
&= 2\epsilon\psi\partial^\mu\phi^* - \epsilon\sigma^\mu\overline{\sigma}^\nu\psi\,\partial_\nu\phi^* + \psi^\dagger\overline{\sigma}^\mu\sigma^\nu\epsilon^\dagger\,\partial_\nu\phi \\
&= \epsilon\sigma^\nu\overline{\sigma}^\mu\psi\,\partial_\nu\phi^* + \psi^\dagger\overline{\sigma}^\mu\sigma^\nu\epsilon^\dagger\,\partial_\nu\phi
\end{aligned}
$$

Results for the supercurrents:

$$
J^\mu_\alpha = (\sigma^\nu\overline{\sigma}^\mu\psi)_\alpha\,\partial_\nu\phi^*\ , \qquad J^{\dagger\mu}_{\dot\alpha} = (\psi^\dagger\overline{\sigma}^\mu\sigma^\nu)_{\dot\alpha}\,\partial_\nu\phi.
$$

The Supercharges

The Noether charge generate the transformations of the corresponding symmetry (see e.g. **PS 9.97** for signs and normalization).

The conserved supercharges:

$$Q_\alpha = \sqrt{2} \int d^3x \, J^0_\alpha \ , \quad Q^\dagger_{\dot\alpha} = \sqrt{2} \int d^3x \, J^{\dagger 0}_{\dot\alpha}$$

generate SUSY transformations when acting on any (string of) fields:

$$\left[\epsilon Q + \epsilon^\dagger Q^\dagger, X\right] = -i\sqrt{2}\, \delta X$$

Commutators of the supercharges acting on fields give:

$$\left[\epsilon_2 Q + \epsilon_2^\dagger Q^\dagger, \, \left[\epsilon_1 Q + \epsilon_1^\dagger Q^\dagger, \, X\right]\right] - \left[\epsilon_1 Q + \epsilon_1^\dagger Q^\dagger, \, \left[\epsilon_2 Q + \epsilon_2^\dagger Q^\dagger, \, X\right]\right]$$
$$= 2(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})X = 2(\epsilon_2 \sigma^\mu \epsilon_1^\dagger - \epsilon_1 \sigma^\mu \epsilon_2^\dagger)\, i\partial_\mu X$$

Reorganize (using the Jacobi identity) and identify $i\partial_\mu = P_\mu$

$$\Big[\big[\epsilon_2 Q + \epsilon_2^\dagger Q^\dagger,\, \epsilon_1 Q + \epsilon_1^\dagger Q^\dagger\big],\, X\Big] = 2(\epsilon_2\sigma^\mu\epsilon_1^\dagger - \epsilon_1\sigma^\mu\epsilon_2^\dagger)\,[P_\mu, X]$$

This is an operator equation, since X is arbitrary

$$\big[\epsilon_2 Q + \epsilon_2^\dagger Q^\dagger,\, \epsilon_1 Q + \epsilon_1^\dagger Q^\dagger\big] = 2(\epsilon_2\sigma^\mu\epsilon_1^\dagger - \epsilon_1\sigma^\mu\epsilon_2^\dagger)\,P_\mu$$

Since ϵ_1 and ϵ_2 are arbitrary:

$$\begin{aligned}\big[\epsilon_2 Q, \epsilon_1^\dagger Q^\dagger\big] &= 2\epsilon_2\sigma^\mu\epsilon_1^\dagger P_\mu \\ \big[\epsilon_2 Q, \epsilon_1 Q\big] &= \big[\epsilon_2^\dagger Q^\dagger, \epsilon_1^\dagger Q^\dagger\big] = 0\end{aligned}$$

Extracting the arbitrary ϵ_1 and ϵ_2:

$$\begin{aligned}&\{Q_\alpha, Q^\dagger_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \\ &\{Q_\alpha, Q_\beta\} = \{Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}\} = 0\end{aligned}$$

This is precisely the SUSY algebra, which is thus realized by the free WZ-model.

The interacting Wess–Zumino model

So far, we considered the free Wess-Zumino model for a single chiral field.

Lagrangian for multicomponent generalization

$$\mathcal{L}_{\text{free}} = \partial^\mu \phi^{*j} \partial_\mu \phi_j + i\psi^{\dagger j} \overline{\sigma}^\mu \partial_\mu \psi_j + \mathcal{F}^{*j} \mathcal{F}_j$$

Offshell SUSY transformations for multicomponent model

$$\begin{aligned} \delta\phi_j &= \epsilon\psi_j & \delta\phi^{*j} &= \epsilon^\dagger \psi^{\dagger j} \\ \delta\psi_{j\alpha} &= -i(\sigma^\mu \epsilon^\dagger)_\alpha \, \partial_\mu \phi_j + \epsilon_\alpha \mathcal{F}_j & \delta\psi^{\dagger j}_{\dot\alpha} &= i(\epsilon\sigma^\mu)_{\dot\alpha} \, \partial_\mu \phi^{*j} + \epsilon^\dagger_{\dot\alpha} \mathcal{F}^{*j} \\ \delta\mathcal{F}_j &= -i\epsilon^\dagger \overline{\sigma}^\mu \partial_\mu \psi_j & \delta\mathcal{F}^{*j} &= i\partial_\mu \psi^{\dagger j} \overline{\sigma}^\mu \epsilon \end{aligned}$$

Next: introduce interactions, while preserving SUSY.

Guiding principle: the off-shell SUSY transformations are independent of interactions.

Interpretation: the SUSY transformations is the realization of the algebra (the same for all interactions), while the interactions specify the e.o.m.'s (not needed at the level of off-shell algebra).

The most general set of renormalizable interactions:

$$\mathcal{L}_{\text{int}} = -\tfrac{1}{2} W^{jk} \psi_j \psi_k + W^j \mathcal{F}_j + h.c.,$$

with W_{jk} linear in ϕ_i, ϕ_i^* and W^j quadratic in ϕ_i, ϕ_i^*.

Recall: $\psi_j \psi_k = \psi_j^\alpha \epsilon_{\alpha\beta} \psi_k^\beta$ is symmetric under $j \leftrightarrow k$. So W^{jk} is symmetric under $j \leftrightarrow k$ (without loss of generality).

Remark: a potential $U(\phi_j, \phi^{*j})$ would break SUSY, since a SUSY transformation gives

$$\delta U = \tfrac{\partial U}{\partial \phi_j} \epsilon \psi_j + \tfrac{\partial U}{\partial \phi^{*j}} \epsilon^\dagger \psi^{\dagger j}$$

which is linear in ψ_j and $\psi^{\dagger j}$ with no derivatives or $\mathcal{F}$ dependence. Such terms cannot be canceled by any other term in $\delta \mathcal{L}_{\text{int}}$.

SUSY Conditions

First focus on variations of $\mathcal{L}_{\rm int}$ with four spinors:

$$\delta\mathcal{L}_{\rm int}|_{4-{\rm spinor}} \;=\; -\frac{1}{2}\frac{\partial W^{jk}}{\partial\phi_n}(\epsilon\psi_n)(\psi_j\psi_k) - \frac{1}{2}\frac{\partial W^{jk}}{\partial\phi^{*n}}(\epsilon^\dagger\psi^{\dagger n})(\psi_j\psi_k) + h.c.$$

The vanishing of the second term requires that W^{jk} is analytic (a.k.a. holomorphic):

$$\frac{\partial W^{jk}}{\partial\phi^{*n}} = 0$$

The Fierz identity

$$(\epsilon\psi_j)(\psi_k\psi_n) + (\epsilon\psi_k)(\psi_n\psi_j) + (\epsilon\psi_n)(\psi_j\psi_k) = 0$$

so the first term vanishes exactly when $\partial W^{jk}/\partial\phi_n$ is totally symmetric under the interchange of j, k, n.

This condition amounts to the existence of a superpotential W such that:

$$W^{jk} = \frac{\partial^2}{\partial\phi_j\partial\phi_k}W$$

Next, the terms in the SUSY variation with derivatives of the fields:

$$\begin{aligned} \delta\mathcal{L}_{\text{int}}|_{\partial} &= -iW^{jk}\partial_\mu\phi_k\,\psi_j\sigma^\mu\epsilon^\dagger - iW^j\,\partial_\mu\psi_j\sigma^\mu\epsilon^\dagger + h.c. \\ &= -i\partial_\mu\left(\frac{\partial W}{\partial\phi_j}\right)\psi_j\sigma^\mu\epsilon^\dagger - iW^j\,\partial_\mu\psi_j\sigma^\mu\epsilon^\dagger + h.c. \end{aligned}$$

This term is a total derivative exactly if

$$W^j = \frac{\partial W}{\partial\phi_j}$$

All the remaining terms in the SUSY variation depend on the auxiliary field:

$$\delta\mathcal{L}_{\text{int}}|_{\mathcal{F},\mathcal{F}^*} = -W^{jk}\mathcal{F}_j\epsilon\psi_k + \frac{\partial W^j}{\partial\phi_k}\epsilon\psi_k\mathcal{F}_j$$

These cancel automatically, if the previous conditions are satisfied.

Summary: the interaction term

$$\mathcal{L}_{\text{int}} = -\frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\partial_j}\psi_i\psi_j + \frac{\partial W}{\partial\phi_i}\mathcal{F}_i + h.c.$$

transforms into a total derivative, for any holomorphic superpotential W.

Remark on Renormalizability

The original *ansatz* for $\mathcal{L}_{\rm int}$ was motivated by renormalizability.

Yet, the computation establishing SUSY did not rely on the functional form of W (other than it must be holomorphic).

For renormalizable interactions

$$W(\phi) = E^i\phi_i + \frac{1}{2}M^{ij}\phi_i\phi_j + \frac{1}{6}y^{ijk}\phi_i\phi_j\phi_k$$

where M^{ij}, y^{ijk} are are symmetric under interchange of indices.

Often we additionally take $E^i = 0$ so SUSY is unbroken in the minimum $\phi_i = 0$.

Integrate out auxillary fields

The action is quadratic in the auxiliary field $\mathcal{F}$ and there are no derivatives:

$$\mathcal{L}_{\mathcal{F}} = \mathcal{F}_j \mathcal{F}^{*j} + W^j \mathcal{F}_j + W_j^* \mathcal{F}^{*j}$$

The path integral can be performed exactly, by solving its algebraic equation of motion:

$$\mathcal{F}_j = -W_j^* \ , \qquad \mathcal{F}^{*j} = -W^j$$

Insert in $\mathcal{L}$:

$$\begin{aligned} \mathcal{L} \quad = \quad & \partial^\mu \phi^{*j} \partial_\mu \phi_j + i\psi^{\dagger j} \overline{\sigma}^\mu \partial_\mu \psi_j \\ & -\tfrac{1}{2} \left(W^{jk} \psi_j \psi_k + W^{*jk} \psi^{\dagger j} \psi^{\dagger k} \right) - W^j W_j^* \end{aligned}$$

Remark: the off-shell SUSY transformation of the fermions ψ_i depended on the auxiliary fields $\mathcal{F}_i$ so, after these are integrated out, it depends on the choice of superpotential W.

WZ Lagrangian

The interacting Wess–Zumino model (with renormalizable superpotential):

$$\begin{aligned}\mathcal{L}_{\rm WZ} \quad = \quad & \partial^\mu \phi^{*j} \partial_\mu \phi_j + i\psi^{\dagger j} \overline{\sigma}^\mu \partial_\mu \psi_j - \tfrac{1}{2} M^{jk} \psi_j \psi_k - \tfrac{1}{2} M^*_{jk} \psi^{\dagger j} \psi^{\dagger k} \\ & - \tfrac{1}{2} y^{jkn} \phi_j \psi_k \psi_n - \tfrac{1}{2} y^*_{jkn} \phi^{*j} \psi^{\dagger k} \psi^{\dagger n} - V(\phi, \phi^*)\end{aligned}$$

The scalar potential:

$$\begin{aligned}& V(\phi, \phi^*) = W^j W^*_j = \mathcal{F}_j \mathcal{F}^{*j} = M^*_{jn} M^{nk} \phi^{*j} \phi_k \\ & + \tfrac{1}{2} M^{jm} y^*_{knm} \phi_j \phi^{*k} \phi^{*n} + \tfrac{1}{2} M^*_{jm} y^{knm} \phi^{*j} \phi_k \phi_n + \tfrac{1}{4} y^{jkm} y^*_{npm} \phi_j \phi_k \phi^{*n} \phi^{*p}\end{aligned}$$

Features:

- The scalar potential

$$V(\phi, \phi^*) = \mathcal{F}_j \mathcal{F}^{*j} \geq 0$$

  as required by SUSY.

- The quartic coupling is $|y|^2$, as required to cancel the Λ^2 divergence in the ϕ-mass.

- The |cubic coupling|^2 $\propto$ quartic coupling $\times |M|^2$ as required to cancel the $\log \Lambda$ divergence.

Linearized equations of motion

Equations of motion, keeping just the quadratic term in the superpotential:

$$\begin{aligned} \partial^\mu\partial_\mu\phi_j &= -M^*_{jn}M^{nk}\phi_k + \ldots; \\ i\overline{\sigma}^\mu\partial_\mu\psi_j &= M^*_{jk}\psi^{\dagger k} + \ldots; \\ i\sigma^\mu\partial_\mu\psi^{\dagger j} &= M^{jk}\psi_k + \ldots \end{aligned}$$

Multiplying fermion equations by $i\sigma^\nu\partial_\nu$ and $i\overline{\sigma}^\nu\partial_\nu$, and using the Pauli identity, we obtain

$$\begin{aligned} \partial^\mu\partial_\mu\psi_j &= -M^*_{jn}M^{nk}\psi_k + \ldots; \\ \partial^\mu\partial_\mu\psi^{\dagger k} &= -\psi^{\dagger j}M^*_{jn}M^{nk} + \ldots \end{aligned}$$

Conclusion: scalars and fermions have the same mass eigenvalues, as required by SUSY.

Diagonalizing the mass matrix gives a collection of massive chiral supermultiplets.